Matrix Diagonalization to Solve a Differential Equation

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Here is a toy problem. Three alien ethnicities have been marooned on a strange planet. They are known as the Bokonists, Cthulhuists, and Spaghettites. The fraction of Bokonists, Cthulhuists, and Spaghettites sum to 1. In each consecutive generation, one fifth of Bokonists will die from despairing suicide. Also, Spaghettites will eat a number of Bokonists equivalent to one fifth of the Spaghettite population size. The stress of the environment has resulted in a negligible Bokonist reproduction rate. There is thus a net loss in Bokonist population. Meanwhile, a net two fifths of Cthulhuists will die in each generation from the dryness of the planet (Cthulhuists are adapted to very wet environments), overwhelming their similarly negligible reproduction rate. Bokonists will also hunt a number of Cthulhuists equivalent to two fifths of the Bokonist population size for medicinal purposes in a hapless attempt to increase their own reproduction rate. Cthulhuists find Spaghettites delicious and eat a number of Spaghettites equivalent to one fifth of the Cthulhuist population in each generation. However, Spaghettites thrive in the terrestrial climate and are taking advantage of new economic opportunities not available in their previous habitat. This is reflected in their net population growth rate of 40 %.

As time goes to infinity, what will the demographics of the planet go towards, given initial population fractions of B_0, C_0, and S_0? We hypothesize intuitively that Spaghettites will dominate and Cthulhuists and Bokonists will go extinct due to their lack of reproduction. To verify this answer more specificially, we can translate the problem, which is a linear system, into the equation

$$\frac{d\mathrm{y}}{dt} = A\mathrm{y}$$

where y is a complementary solution of the linear sytem and

$$A = \begin{bmatrix} -0.2 & 0 & -0.2 \\ -0.4 & -0.4 & 0 \\ 0 & -0.2 & 0.4 \end{bmatrix}$$

To solve for a complementary solution y, we can use a technique called matrix diagonalization to decouple this linear system. We substitute y for a z where

$$\begin{aligned} \mathrm{y} &= P\mathrm{z} \\ \mathrm{z} &= P^{-1}\mathrm{y} \end{aligned}$$

P is a matrix consisting of the eigenvectors of A. The relationship between y and z comes from the fact that $A = PDP^{-1}$, where D is a diagonal matrix of the eigenvalues of A.

This lets us solve for the decoupled system

$$\mathrm{z}' = D\mathrm{z}$$

In this case, solving for the eigenvalues with software like Mathematica or Matlab (or something open source),

$$D = \begin{bmatrix} -0.469 & 0 & 0 \\ 0 & 0.363 & 0 \\ 0 & 0 & -0.0941 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.165 & -0.33 & -0.578 \\ 0.961 & 0.173 & 0.756 \\ 0.221 & 0.928 & 0.306 \end{bmatrix}$$

With 3 by 3 matrices, we are working with a system that is small enough to be proceed by the superposition principle. However, with larger matrices, an alternate method involving integration by parts may be more effective. We will use both methods.

$$\mathrm{z}' = \begin{bmatrix} -0.469 & 0 & 0 \\ 0 & 0.363 & 0 \\ 0 & 0 & -0.0941 \end{bmatrix} \mathrm{z}$$

$$\begin{aligned} \mathrm{z}_1' + 0.469\mathrm{z}_1 &= 0 \\ \mathrm{z}_2' - 0.363\mathrm{z}_1 &= 0 \\ \mathrm{z}_3' + 0.0941\mathrm{z}_1 &= 0 \end{aligned}$$

Invoking the integration by parts technique, we multiply the equations by integrating factors $e^{\lambda t}$.

$$\begin{aligned} e^{0.469t}(\mathrm{z}_1' + 0.469\mathrm{z}_1) &= 0 \\ e^{-0.363t}(\mathrm{z}_2' - 0.363\mathrm{z}_1) &= 0 \\ e^{0.0941t}(\mathrm{z}_3' + 0.0941\mathrm{z}_1) &= 0 \end{aligned}$$

And then integrating, (I skip several steps in the middle)

$$\begin{aligned} \mathrm{z}_1 &= C_1 e^{-0.469t} \\ \mathrm{z}_2 &= C_2 e^{0.363t} \\ \mathrm{z}_3 &= C_3 e^{-0.0941t} \end{aligned}$$

where C_1, C_2, and C_3 are constants that depend on the initial values B_0, C_0, and S_0.

As y = Pz,

$$y(t) = \begin{bmatrix} 0.165 & -0.33 & -0.578 \\ 0.961 & 0.173 & 0.756 \\ 0.221 & 0.928 & 0.306 \end{bmatrix} \begin{bmatrix} C_1 e^{-0.469t} \\ C_2 e^{0.363t} \\ C_3 e^{-0.0941t} \end{bmatrix}$$

From there, you can multiply the matrices and then solve for C_1, C_2, and C_3.

It can useful to go through both superposition and integration by parts methods for a 2 by 2 or 3 by 3 matrix to check solutions. Invoking the superposition principle for our aforementioned system, we come up with the equation

$$y(t) = C_1 \begin{bmatrix} 0.165 \\ 0.961 \\ 0.221 \end{bmatrix} e^{-0.469t} + C_2 \begin{bmatrix} -0.33 \\ 0.173 \\ 0.928 \end{bmatrix} e^{0.363t} + C_3 \begin{bmatrix} -0.578 \\ 0.756 \\ 0.306 \end{bmatrix} e^{-0.0941t}$$

where C_1, C_2, and C_3 are constants that depend on the initial population propotions B_0, C_0, and S_0. We can see that as $t \to \infty$, the equation becomes

$$y(t) = C_2 \begin{bmatrix} -0.33 \\ 0.173 \\ 0.928 \end{bmatrix} e^{0.363t}$$

with the other two terms going to 0.

If we multiply Pz, we can see that the equation generated from the integration by parts and the superposition methods are equivalent.